

## UNIT-REGULARITY OF REGULAR NILPOTENT ELEMENTS

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ABSTRACT. Let  $a$  be a regular element of a ring  $R$ . If either  $K := r_R(a)$  has the exchange property or every power of  $a$  is regular, then we prove that for every positive integer  $n$  there exist decompositions

$$R_R = K \oplus X_n \oplus Y_n = E_n \oplus X_n \oplus aY_n,$$

where  $Y_n \subseteq a^n R$  and  $E_n \cong R/aR$ . As applications we get easier proofs of the results that a strongly  $\pi$ -regular ring has stable range one and also that a strongly  $\pi$ -regular element whose every power is regular is unit-regular.

An element  $a$  of a ring  $R$  is called strongly  $\pi$ -regular if both chains  $aR \supseteq a^2R \supseteq a^3R \dots$  and  $Ra \supseteq Ra^2 \supseteq Ra^3 \dots$  stabilize. If every element of  $R$  is strongly  $\pi$ -regular, then  $R$  is called a *strongly  $\pi$ -regular ring*. In [1] Pere Ara proved a wonderful result that a strongly  $\pi$ -regular ring has stable range one. Ara's proof is on the following lines. As a strongly  $\pi$ -regular ring is an exchange ring and an exchange ring has stable range one if and only if every regular element is unit-regular, it is enough to show that every regular element of a strongly  $\pi$ -regular ring is unit-regular. Suppose  $a$  is a regular element of a strongly  $\pi$ -regular ring. By [7, Proposition 1] there exist  $n \in \mathbb{N}$ , an idempotent  $e$  and a unit  $u$  in  $R$  with  $a^n = eu$  such that  $a$ ,  $e$  and  $u$  commute with each other. Then  $ea$  is a unit in  $eRe$  with inverse  $ea^{n-1}u^{-1}$  and  $(1-e)a$  is a regular nilpotent element of the exchange ring  $(1-e)R(1-e)$ . As  $a = ea + (1-e)a$  and  $ea$  is unit-regular in  $eRe$ , we will get that  $a$  is unit-regular if we can show that  $(1-e)a$  is unit-regular in  $(1-e)R(1-e)$ . So the result will follow if we can show that a regular nilpotent element of an exchange ring is unit-regular. This is the crucial result proved by Ara in [1] and an easier proof of this will follow from our Theorem 2.

In [5, Theorem 5.8] Goodearl and Menal proved that a regular strongly  $\pi$ -regular ring is unit-regular. The proof of [5, Theorem 5.8] can be adapted to prove that if  $a$  is a strongly  $\pi$ -regular element of any ring  $R$  such that  $a^n$  is regular for every  $n \in \mathbb{N}$ , then  $a$  is unit-regular. A different proof of this result was given by Beidar, O'Meara and Raphael in [3, Corollary 3.7]. Suppose  $a$  is a strongly  $\pi$ -regular element of a ring  $R$  whose each power is regular. Then as above there exist  $n \in \mathbb{N}$ , an idempotent  $e$  and a unit  $u$  in  $R$  with  $a^n = eu$  such that  $a$ ,  $e$  and  $u$  commute with each other. As seen above it will follow that  $a$  is unit-regular if we can prove that  $(1-e)a$  is unit-regular in  $(1-e)R(1-e)$ . As  $(1-e)a$  is nilpotent and its each power is regular in  $(1-e)R(1-e)$ , it is enough to prove that a nilpotent element whose each power is regular is unit-regular. An easier proof of this will follow from our Theorem 4.

Recently Ara and O'Meara in [2] and Pace and Šter in [8] have shown that a regular nilpotent element in general may not be unit-regular.

By  $A \subseteq^\oplus B$  we shall mean that  $A$  is a summand of the module  $B$ . We will tacitly use the fact that a regular element  $a \in R$  is unit-regular if and only if  $r_R(a) \cong R/aR$ , where  $r_R(a) = \{x \in R : ax = 0\}$ .

**Lemma 1** [4, Corollary 3.9]. *If  $M$  has the exchange property and  $A = M \oplus B \oplus C = \bigoplus_I A_i \oplus C$ , then there exists a decomposition  $A_i = D_i \oplus E_i$  of each  $A_i$  such that  $A = M \oplus \bigoplus_I D_i \oplus C$ .*

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**Theorem 2.** *Let  $a$  be a regular element of a ring  $R$  such that the right  $R$ -module  $K := r_R(a)$  has the exchange property. Then for every  $n \in \mathbb{N}$  there exist decompositions*

$$R_R = K \oplus X_n \oplus Y_n = E_n \oplus X_n \oplus aY_n,$$

where  $Y_n \subseteq a^n R$  and  $E_n \cong R/aR$ . If  $a^n = 0$ , then  $aY_n = Y_n = 0$  and so  $K \cong E_n \cong R/aR$  implying that  $a$  is unit-regular.

**Proof.** For every positive integer  $i$  we will inductively construct right ideals  $A_i, A'_i, Y_i$  of  $R$  such that for every  $j \geq 1$ ,

$$R = K \oplus \left( \bigoplus_{i=1}^j A_i \right) \oplus Y_j = \left( \bigoplus_{i=1}^j A_i \right) \oplus (A'_j \oplus aA_j) \oplus aY_j, \quad (*)$$

where  $Y_j \subseteq a^j R$  and  $A'_j \oplus aA_j = A_{j+1} \oplus A'_{j+1} \cong R/aR$ . Then we have the desired decompositions by putting  $X_n = \bigoplus_{i=1}^n A_i$  and  $E_n = A'_n \oplus aA_n$ .

We have  $R = K \oplus B = A \oplus aR$  for some right ideals  $A$  and  $B$  of  $R$ . As  $K_R$  has the exchange property, by Lemma 1 we have decompositions  $A = A_1 \oplus A'_1$  and  $aR = Y_1 \oplus Y'_1$  such that  $R = K \oplus A_1 \oplus Y_1$ . As  $K \cap (A_1 \oplus Y_1) = 0$ ,  $aR = aA_1 \oplus aY_1$  and  $aA_1 \cong A_1$ . So  $R = A_1 \oplus A'_1 \oplus aR = A_1 \oplus A'_1 \oplus aA_1 \oplus aY_1$  implying that

$$R = K \oplus A_1 \oplus Y_1 = A_1 \oplus (A'_1 \oplus aA_1) \oplus aY_1,$$

where  $Y_1 \subseteq aR$  and  $A'_1 \oplus aA_1 \cong A'_1 \oplus A_1 = A \cong R/aR$ .

Now suppose we have found the right ideals  $A_i, A'_i, Y_i$  for  $i = 1, \dots, n$  such that  $(*)$  holds for every  $j = 1, \dots, n$  with  $Y_i \subseteq a^i R$ ,  $A'_i \oplus aA_i \cong R/aR$  for every  $i = 1, \dots, n$  and  $A'_i \oplus aA_i = A'_{i+1} \oplus A_{i+1}$  for every  $i = 1, \dots, n-1$ . As  $K$  has the exchange property and

$$R = K \oplus \left( \bigoplus_{i=1}^n A_i \right) \oplus Y_n = \left( \bigoplus_{i=1}^n A_i \right) \oplus (A'_n \oplus aA_n) \oplus aY_n,$$

by Lemma 1 we have decompositions  $A'_n \oplus aA_n = A_{n+1} \oplus A'_{n+1}$  and  $aY_n = Y_{n+1} \oplus Y'_{n+1}$  such that

$$R = K \oplus \left( \bigoplus_{i=1}^{n+1} A_i \right) \oplus Y_{n+1}.$$

So  $aR = \bigoplus_{i=1}^{n+1} aA_i \oplus aY_{n+1}$  and  $aA_{n+1} \cong A_{n+1}$ . Now as  $A'_j \oplus aA_j = A_{j+1} \oplus A'_{j+1}$  for every  $j = 1, \dots, n$  we have

$$R = A_1 \oplus A'_1 \oplus aR = A_1 \oplus A'_1 \oplus \bigoplus_{i=1}^{n+1} aA_i \oplus aY_{n+1} = \left( \bigoplus_{i=1}^{n+1} A_i \right) \oplus (A'_{n+1} \oplus aA_{n+1}) \oplus aY_{n+1},$$

with  $Y_{n+1} \subseteq a^{n+1} R$  and  $A'_{n+1} \oplus aA_{n+1} \cong A'_{n+1} \oplus A_{n+1} = A'_{n+1} \oplus aA_n \cong R/aR$ .  $\square$

**Lemma 3** [6, Lemma 2.8]. *Let  $P_R$  be a projective module and  $P = A + B$  where  $A \subseteq^\oplus P$ . Then  $B = C \oplus D$  for some submodules  $C$  and  $D$  such that  $P = A \oplus C$ .*

**Theorem 4.** *Let  $a$  be an element of a ring  $R$  such that  $a^n$  is regular for every positive integer  $n$ . Then for every  $n \in \mathbb{N}$  there exist decompositions*

$$R_R = K \oplus X_n \oplus Y_n = E_n \oplus X_n \oplus aY_n,$$

where  $K = r_R(a)$ ,  $Y_n \subseteq a^n R$ ,  $K \oplus Y_n = K + a^n R$  and  $E_n \cong R/aR$ . If  $a^n = 0$ , then  $K \cong E_n \cong R/aR$  implying that  $a$  is unit-regular.

**Proof.** For every positive integer  $i$  we will inductively construct right ideals  $A_i$ ,  $A'_i$ ,  $Y_i$  of  $R$  such that for every  $j \geq 1$ ,

$$R = K \oplus \left( \bigoplus_{i=1}^j A_i \right) \oplus Y_j = \left( \bigoplus_{i=1}^j A_i \right) \oplus (A'_j \oplus aA_j) \oplus aY_j, \quad (**)$$

where  $Y_j \subseteq a^j R$ ,  $aY_j = a^{j+1}R$  and  $A'_j \oplus aA_j = A_{j+1} \oplus A'_{j+1} \cong R/aR$ . Then we have the desired decompositions by putting  $X_n = \bigoplus_{i=1}^n A_i$  and  $E_n = A'_n \oplus aA_n$ .

Note that the left multiplication by  $a$  induces an epimorphism from  $R \rightarrow aR/a^{n+1}R$  with kernel  $K + a^n R$ . As  $a$  and  $a^{n+1}$  are regular,  $aR/a^{n+1}R$  is projective implying that  $K + a^n R \subseteq^\oplus R_R$  for each  $n$ . By Lemma 3,  $K + aR = K \oplus Y_1$  for some  $Y_1 \subseteq aR$ . If  $Y'_1 = aR \cap K$ , then on intersecting with  $aR$  we have  $aR = Y'_1 \oplus Y_1$ ,  $a^2R = aY_1$  and  $Y'_1 \subseteq^\oplus R_R$ . So  $K = Y'_1 \oplus A'_1$  for some  $A'_1$ . Also for some right ideal  $A_1$  we have  $R = (K + aR) \oplus A_1 = K \oplus Y_1 \oplus A_1 = Y'_1 \oplus A'_1 \oplus Y_1 \oplus A_1$ . As  $K \cap (A_1 \oplus Y_1) = 0$ ,  $aR = aA_1 \oplus aY_1$  and  $aA_1 \cong A_1$ . So  $R = A_1 \oplus A'_1 \oplus aR = A_1 \oplus A'_1 \oplus aA_1 \oplus aY_1$ . Thus

$$R = K \oplus A_1 \oplus Y_1 = A_1 \oplus (A'_1 \oplus aA_1) \oplus aY_1,$$

where  $Y_1 \subseteq aR$ ,  $K \oplus Y_1 = K + aR$  and  $A'_1 \oplus aA_1 \cong A'_1 \oplus A_1 \cong R/aR$ .

Now suppose we have found the right ideals  $A_i$ ,  $A'_i$ ,  $Y_i$  for  $i = 1, \dots, n$  such that we have decompositions as in (\*\*) for every  $j = 1, \dots, n$  with  $Y_i \subseteq a^i R$ ,  $K \oplus Y_i = K + a^i R$ ,  $A'_i \oplus aA_i \cong R/aR$  for every  $i = 1, \dots, n$  and  $A'_i \oplus aA_i = A'_{i+1} \oplus A_{i+1}$  for every  $i = 1, \dots, n-1$ . By Lemma 3,  $K + a^{n+1}R = K \oplus Y_{n+1}$  for some  $Y_{n+1} \subseteq a^{n+1}R$ . Note that  $aY_n = a^{n+1}R \subseteq K \oplus Y_{n+1}$  and  $(K \oplus Y_{n+1}) \cap \bigoplus_{i=1}^n A_i \subseteq (K \oplus Y_n) \cap \bigoplus_{i=1}^n A_i = 0$ . As  $K + a^n R \subseteq^\oplus R_R$  for each  $n$ ,  $K \oplus Y_{n+1} = K + a^{n+1}R \subseteq^\oplus K + a^n R = K \oplus Y_n$  and  $K \oplus Y_{n+1} \oplus \bigoplus_{i=1}^n A_i \subseteq^\oplus K \oplus Y_n \oplus \bigoplus_{i=1}^n A_i = R$ . Thus

$$R = (K \oplus Y_{n+1}) + R = (K \oplus Y_{n+1}) + \left( \bigoplus_{i=1}^n A_i \oplus (A'_n \oplus aA_n) \oplus aY_n \right) = (K \oplus Y_{n+1} \oplus \bigoplus_{i=1}^n A_i) + (A'_n \oplus aA_n).$$

Again by Lemma 3,  $A'_n \oplus aA_n = A_{n+1} \oplus A'_{n+1}$  such that  $R = K \oplus Y_{n+1} \oplus \bigoplus_{i=1}^{n+1} A_i$ . So  $aR = \bigoplus_{i=1}^{n+1} aA_i \oplus aY_{n+1}$ ,  $aA_{n+1} \cong A_{n+1}$ . Now as  $A'_j \oplus aA_j = A_{j+1} \oplus A'_{j+1}$  for every  $j = 1, \dots, n$  we have

$$R = A_1 \oplus A'_1 \oplus aR = A_1 \oplus A'_1 \oplus \bigoplus_{i=1}^{n+1} aA_i \oplus aY_{n+1} = \left( \bigoplus_{i=1}^{n+1} A_i \right) \oplus (A'_{n+1} \oplus aA_{n+1}) \oplus aY_{n+1},$$

with  $Y_{n+1} \subseteq a^{n+1}R$ ,  $K \oplus Y_{n+1} = K + a^{n+1}R$  and  $A'_{n+1} \oplus aA_{n+1} \cong A'_{n+1} \oplus A_{n+1} = A'_n \oplus aA_n \cong R/aR$ .  $\square$

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